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Asymptotics for Angelesco–Nikishin Systems

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Weak asymptotics for simultaneous Hermite–Pade approximants are obtained for a system of Markov-type functions.

1. INTRODUCTION

Let $(f_1, ..., f_m)$ be a vector of complex-valued functions f_i $(i=1, ..., m)$ analytic in a neighborhood of infinity. For each natural n we can select a vector of rational functions

$$(Q_1/Q_n, ..., Q_m/Q_n)$$

with polynomials of degree not greater than n such that, for $i=1, ..., m$, Q_i/Q_n interpolates f_i at infinity with a degree as high as possible. What can we say about convergence of these simultaneous rational approximants? It is very difficult to give an answer without more special assumptions about the vector. Here we consider a case when each f_i is a Markov function.

1.1. *AN-Systems.* Let us define a particular system. Let m and b be positive fixed integers,

$$a = mb.$$

Suppose for $i=1, ..., m$, $j=1, ..., b$, $\{\mu_{i,j}\}$ is a set of positive Borel measures with compact support in $\mathbf{R}$ and assume (to exclude trivial cases) that $\operatorname{supp}(\mu_{i,j})$ is an infinite set. By $\Delta_{i,j}$ we denote the convex hull of $\operatorname{supp}(\mu_{i,j})$, $D_i = \mathbf{C} \backslash \Delta_{i,1}$, and $\mu'_{i,j}$ denotes the Radon–Nikodym derivative of $\mu_{i,j}$. Some restrictions on $\Delta_{i,j}$ are needed:

$$\text{I.} \qquad \Delta_{i,1} \cap \Delta_{j,1} = \varnothing, \qquad i \neq j. \tag{1.1}$$

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II. If $b>1$ then

$$\Delta_{i,j} \cap \Delta_{i,j+1} = \varnothing, \qquad j=1, ..., (b-1).$$

For fixed i ($i=1, ..., m$) define

$$g_{i,j+1,j}(z) = 1, \qquad j=1, ..., b,$$

$$g_{i,b,b}(z) = \int_{\Delta_{i,b}} \frac{d\mu_{i,b}(x)}{z-x}, \qquad z \in C \backslash \Delta_{i,b},$$

and if $b>1$, (1.2)

$$g_{i,j,k}(z) = \int_{\Delta_{i,j}} \frac{g_{i,j+1,k}(x)\, d\mu_{i,j}(x)}{z-x}, \qquad z \in C \backslash \Delta_{i,j}$$

$j=1, ..., b-1$, $k=j, ..., b$.

Definition 1. The family $\{g_{i,1,k} : i=1, ..., m,\ k=1, ..., b\}$ where $g_{i,1,k}$ is given by (1.1), (1.2) is called an AN-system determined by the measures $(\mu_{i,j})$.

Since for fixed i the functions $g_{i,j,k}$ are defined recursively the order of numeration is not arbitrary. If for an AN-system $m=1$ and $b=1$, then we have a simple Markov function. If $m=1$ and $b>1$, the AN-system is a Nikishin system (of order b) on $\Delta_{1,1}$. Such systems were introduced and studied by Nikishin in [Ni]; further properties can be found in [Ni; NiSo, Chap. 4.4; Bulo2; Pi; and DriSt]. If $m>1$ and $b=1$, the AN-system is an Angelesco system; see [A]. In [GoRa1] some general results were obtained on the convergence of simultaneous approximants for the last systems. In the notation above A stands for Angelesco and N for Nikishin. In this paper we work with a general AN-system, and assume that all measures $\mu_{i,j}$ are given and fixed as above. Note that for fixed i, the system of functions

$$g_{i,1,k}, \qquad k=1, 2, ..., b \qquad (b>1)$$

is a Nikishin system.

A family $\{u_k\}_{k=0}^{n}$, $u_k \in C[\Delta]$ (Δ a bounded closed interval), is called a Chebisheff system of order n if for any reals α_k ($k=0, 1, ..., n$), $\sum_{k=0}^{n} \alpha_k^2 > 0$, the function

$$\sum_{k=0}^{n} \alpha_k u_k(x)$$

has no more than n zeros in Δ.

It is known that if $f_1, ..., f_r$ is a Nikishin system (of order r) on Δ and $n_0 \geqslant n_1 \geqslant \cdots \geqslant n_r$ then

$$1, ..., x^{n_0-1}, f_1(x), ..., x^{n_1-1} f_1(x), ..., f_r(x), ..., x^{n_r-1} f_r(x)$$

is a Chebisheff system of order

$$n_0 + n_1 + \cdots + n_r - 1.$$

Moreover, if $n_0 + n_1 + \cdots + n_r - 1$ distinct points are given in $\operatorname{int}(\Delta)$, there exist polynomials P_k, $\deg P_k \leqslant n_k - 1$, such that

$$\sum_{k=0}^{n} P_k(x)\, f_k(x) \qquad (f_0 = 1)$$

has a simple zero at each one of the fixed points and has no other zeros in $\operatorname{int}(\Delta)$ (see [NiSo, Chap. IV.4, Theorem 4.4, and Corollary]). The same result remains under the more general condition

$$n_k \geqslant -1 + \max\{n_{k+1}, n_{k+2}, ..., n_r\}$$

for $k = 0, ..., (r-1)$ (see [DrSt]). This fact will be used below.

1.2. *Some Notation*

In the sequel $\Pi(n)$ denotes the set of all polynomials P, $\deg P \leqslant n$, $n \in N$. $\Pi^*(n)$ is the set of all monic polynomials in $\Pi(n)$.

For each n $(n > a)$, i $(i = 1, ..., m)$, and j $(1 \leqslant j \leqslant b)$, we fix positive integers $n(i, j)$ such that

I. if $b > 1$,

$$n(i, j) \geqslant -1 + \max\{n(i, j+1), ..., n(i, b)\},$$

for $j = 1, ..., b-1$;

II. $$n = \sum_{i=1}^{m} \sum_{k=1}^{b} n(i, k). \tag{1.3}$$

III. For fixed i $(= 1, ..., m)$ and k $(= 1, ..., b)$,

$$\frac{n(i, k)}{n} \to \frac{1}{a}.$$

1.3. *Simultaneous Hermite-Pade Approximants for AN-Systems*

Given $n \in N$ $(n > a)$, we seek polynomials $Q_n \in \Pi(n)$, $Q_n \neq 0$, and $P_{n,i,k} \in \Pi(n)$, for $i = 1, ..., m$, $1 \leqslant k \leqslant b$, such that

$$F_{n,i,k}(z) := Q_n(z)\, g_{i,1,k}(z) - P_{n,i,k}(z) = O(z^{-1-n(i,k)}) \tag{1.4}$$

as $z \to \infty$. $P_{n,i,k}$ is the principal part of $Q_n g_{i,n,k}$ at infinity and O denotes Landau's big oh.

It is easy to show that the above problem has a nontrivial solution. Indeed, conditions (1.4) yield

$$\sum_{i=1}^{m} \sum_{k=1}^{b} (n(i,k)+n+1) = (n+1)a+n$$

homogeneous linear equations in the

$$1+n+\sum_{i=1}^{m} \sum_{k=1}^{b} (n+1) = (n+1)(a+1)$$

unknown coefficients of Q_n and $P_{n,i,k}$. We prove below that for any nontrivial solution $\deg Q_n = n$, then it we take $Q_n \in \Pi^*(n)$ the polynomials $P_{n,i,k}$ are uniquely determined by (1.4). The polynomials $P_{n,i,k}$ introduced in (1.4) are often called Hermite–Pade polynomials of type II. For a review on the (strong and weak) asymptotic behavior of Hermite–Pade polynomials see [ApSt].

Definition 2. Let Q_n, $P_{n,i,k}$ be given as above ($Q_n \in \Pi^*(n)$). The vector

$$\left(\frac{P_{n,1,1}}{Q_n}, ..., \frac{P_{n,1,b}}{Q_n}, \frac{P_{n,2,1}}{Q_n}, ..., \frac{P_{n,m,b}}{Q_n}\right)$$

is called a simultaneous rational (or Hermite–Pade) approximant to the vector

$$(g_{1,1,1}, ..., g_{1,1,b}, g_{2,1,1}, ..., g_{m,1,b}).$$

Remarks. If $m=1$ and $b=1$ we have the (n,n) Pade approximant to a single Markov function ($g = g_{1,1,1}$). From Markov's classical theorem we know that, in such a case, $F_{n,1,1}/Q_n$ converges uniformly to zero on each compact subset of D_1, briefly

$$\frac{P_{n,1,1}}{Q_n} \to g_{1,1,1}, \qquad \text{inside } D_1.$$

Moreover,

$$\overline{\lim_n} \left| g_{1,1,1}(z) - \frac{P_{n,1,1}}{Q_n}(z) \right|^{1/n} \leqslant e^{-2g(z,\infty)}, \tag{1.5}$$

where $g(z, \infty)$ is Green's function of the domain D_1. If $\mu'_{1,1} > 0$ a.e. on Δ_1, we have equality and a regular limit in (1.5).

In [GoRa1] the convergence of the simultaneous approximants for an Angelesco system was studied. They obtained results analogous to (1.5) (with $\mu'_{i,j} > 0$ a.e.) and proved that, in general, we cannot expect convergence of the approximant in the largest possible region.

In [BuLa1] the convergence of the Hermite–Pade approximants for a Nikishin system was studied ($m=1$, $b>1$) without any additional assumption on $\mu_{i,j}$. Here, we always have convergence of the rational approximants in the largest possible region ($n(1, 1) - n(1, b) \leqslant 1$).

Here we present some new results for AN-systems combining ideas of [GoRa1] and [BuLo1]. The first step is to find a function $H(z)$ such that

$$\lim_n |Q_n(z)|^{1/n} = H(z).$$

To this end we follow the potential-theoretic approach employed in [GoRa1]. We need some extremal relations. We will introduce some auxiliary functions which allow us to show that in (1.4) some extra interpolation points appear and the polynomials Q_n may be factorized into orthogonal polynomials with respect to some varying measures. To do it, no additional hypotheses on the measures $\mu_{i,j}$ are needed, but for reasons connected with the solution of the corresponding potential theoretic problem, we assume that $\mu'_{i,j} > 0$ a.e. The existence of extra interpolation points was used in [BuLo1] to prove convergence of the approximants for a general Nikishin system.

1.4. *Statement of the Main Results*

The set of all unit (finite) positive Borel measures μ, such that $\operatorname{supp} \mu \subset \Delta$ (Δ is a fixed bounded closed interval of the real line $\mathbf{R}$) is denoted by $M(\Delta)$ and $M_d(\Delta)$ is the set of all measures λ of the form

$$\lambda = \frac{1}{d} \sum_{k=1}^{d} \delta_{t_k} \tag{1.6}$$

where $t_1, ..., t_d$ are points in Δ (not necessarily distinct) and δ_t is the unit measure concentrated at point t. We can associate to λ the polynomial

$$p(x) = \prod_{k=1}^{d} (x - t_k) \in \Pi^*(d),$$

and reciprocally to each polynomial in $\Pi^*(d)$ with all its zeros in Δ there corresponds a unique measure in $M_d(\Delta)$.

The (logarithmic) potential of a measure $\lambda \in M(\Delta)$ is denoted by

$$V_\lambda(z) := \int_\Delta \log \frac{1}{|z-t|}\, d\lambda(t), \qquad z \in C.$$

The symbol $\lambda_n \to \lambda$, applied to sequences of measures, stands for weak convergence. It is known that $M(\Delta)$ is weakly compact (for this and other results on potential theory see [Lan], also [NiSo]). Therefore, each sequence $\{\lambda_n\}$ in $M(\Delta)$ contains a subsequence $\{\lambda_{n_s}\}$ such that $\lambda_{n_s} \to \lambda$, $\lambda \in M(\Delta)$. For the intervals Δ considered below,

$$\varphi_n \to \varphi, \qquad (\varphi_n,\ \varphi \in M(\Delta)) \Leftrightarrow V_{\varphi_n} \to V_\varphi \text{ inside } C\backslash\Delta.$$

Our main result is the following

THEOREM 1. *Suppose that* $\mu'_{i,j} > 0$ *a.e.* $(i = 1, ..., m;\ j = 1, ..., b)$, *then there exist measures* $\varphi_{i,0} \in M(\Delta_{i,1})$ *such that*

$$\lim |Q_n(z)|^{1/n} = \exp\left(-\frac{1}{m} \sum_{i=1}^{m} V_{\varphi_{i,0}}(z)\right),$$

where Q_n *is given by* (1.4).

As can be seen from the Proof of Theorem 1, the measures $\varphi_{i,0}$ are determined as the (unique) solution of an extremal problem of potential theory.

In regards to the convergence of Hermite–Pade approximants, we have

THEOREM 2. *Under assumptions of Theorem* 1, *for* $i = 1, ..., m$ *there exist measures* $\varphi_{i,0} \in M(\Delta_{i,1})$ *and* (*if* $b > 1$) $\varphi_{i,1} \in M(\Delta_{i,2})$ *such that*:

(a) *If* $b > 1$

$$\lim \left| g_{i,1,1}(z) - \frac{P_{n,i,1}(z)}{Q_n(z)} \right|^{1/n}$$

$$= \exp \frac{1}{m}\left(-\frac{b-1}{b} V_{\varphi_{i,1}}(z) + \sum_{k=0}^{m} V_{\varphi_{k,0}}(z) + V_{\varphi_{i,0}}(z) - m_{i,1}\right),$$

where

$$m_{i,1} = \min_{x \in \Delta_{i,1}} \left(\sum_{k=0}^{m} V_{\varphi_{k,0}}(x) - \frac{b-1}{b} V_{\varphi_{i,1}}(x) + V_{\varphi_{i,0}}\right)(x).$$

(b) *If* $b=1$ *then*

$$\lim \left| g_{i,1,1}(z) - \frac{P_{n,i,1}(z)}{Q_n(z)} \right|^{1/n} = \exp \frac{1}{m} \left(\sum_{k=1}^{m} V_{\varphi_{k,0}}(z) + V_{\varphi_{i,0}}(z) - m_{i,0} \right),$$

where

$$m_{i,0} = \min_{x \in \Delta_{i,0}} \left(\sum_{k=0}^{m} V_{\varphi_{k,0}}(x) + V_{\varphi_{i,0}}(x) \right).$$

The measures $\varphi_{i,0}$ are the same in Theorems 1 and 2. Note that for $m>1$ and $b=1$, Theorems 1 and 2 reduce to the main statements in [GoRa1] if we take $c_i = 1/m$. Theorem 2 gives the asymptotic for the first function on each interval but nothing is said about $(g_{i,1,k} - P_{n,1,k}/Q_n)$ when $1 < k \leqslant b$. To obtain the corresponding asymptotic, new extremal relations are needed. For this purpose, we need to find new interpolation relations. We will illustrate the method discussing a particular system.

THEOREM 3. *Let* $m=2$, $b=2$, $\Delta_{1,2} = [c_1, d_1]$, *and* $\Delta_{2,2} = [c_2, d_2]$ $(c_1 < d_1 < c_2 < d_2)$, *and suppose that* $g_{1,2,2}$ *and* $g_{2,2,2}$ *have finite limits* $g_{1,2,2}(c_1^-)$, $g_{1,2,2}(d_1^+)$, $g_{2,2,2}(c_2^-)$, *and* $g_{2,2,2}(d_2^+)$. *Take* $n(1,2) < n(1,1)$ *and* $n(2,2) < n(2,1)$, $n = n(1,1) + n(1,2) + n(2,1) + n(2,2)$.

Then there exist measures $\sigma_1 \in M(\Delta_{1,2})$, $\sigma_2 \in M(\Delta_{2,2})$ *and constants* α_1, α_2, β_1, β_2, m_1, *and* m_2 *such that*

$$\lim_n |(\alpha_k z + \beta_k) F_{n,k,1}(z) - F_{n,k,2}(z)|^{1/n} = \exp \tfrac{1}{4}(2V_{\varphi_{k,0}}(z) - V_{\sigma_k}(z) - m_k),$$

$$k = 1, 2,$$

where $\varphi_{k,0}$ *is given as in Theorems* 1 *and* 2, *and* σ_k *is the* (*unique*) *solution of the extremal problem*

$$\min_{x \in \Delta_{k,2}} (-V_{\varphi_{k,0}}(x) + V_{\sigma_k}(x)) = \max_{\lambda \in M(\Delta_{k,2})} \min_{x \in \Delta_{k,2}} (-V_{\varphi_{k,0}}(x) + 2V_\lambda(x)).$$

Theorem 3 will be proved in Section 4. This gives indirect information about the asymptotic behavior of $F_{n,1,2}$ and $F_{n,2,2}$. The restriction in function $g_{1,2,2}$ and $g_{2,2,2}$ is due to the fact that we need $F_{n,2,2}$ and $F_{n,1,2}$ to satisfy some orthogonality relations. In proving Theorem 2, we observe that an extra amount of interpolation points automatically appear on segment $\Delta_{2,1}$ for each one of the functions $F_{n,1,1}$ and $F_{n,2,1}$. For the function $F_{n,1,1}$ it is sufficient to infer some extremal relations (see Section 2 below) that allows us to continue the proof as in [BuLo1]. But it is expected that for general selections of the indexes $n(1,1)$, $n(1,2)$, $n(2,1)$, and $n(2,2)$ there are not sufficient extra interpolation points for $F_{n,1,2}$ in $\Delta_{2,1}$.

In Theorems 1 and 2 we study a symmetrical case, that is, for $i=1, ..., m$ we have on $\Delta_{i,1}$ a Nikishin system of order b. It is possible to obtain an extension to nonsymmetrical cases.

2. Proofs of the Main Theorems

First, let us obtain some auxiliary formulas (for the proofs of the lemmas see Section 3).

Lemma 2.1. *For a fixed i $(i=1, ..., m)$*

$$\text{(a)} \quad \int_{\Delta_{i,1}} (Q_n g_{i,2,k} P)(x)\, d\mu_{i,1}(x) = 0, \qquad k=1, ..., b,$$

where $P \in \Pi(n(i,k)-1)$ is an arbitrary polynomial.

$$\text{(b)} \quad (PF_{n,i,k})(z) = \int_{\Delta_{i,1}} \frac{(Q_n g_{i,2,k} P)(x)}{z-x}\, d\mu_{i,1}(x), \qquad k=1, ..., b,$$

where $P \in \Pi(n(i,k))$ is an arbitrary polynomial.

We will use Lemma 2.1 to obtain some information on the location of the zeros of Q_n. Set

$$\alpha(n,i,j) := \sum_{k=j+1}^{b} n(i,k).$$

Lemma 2.2. $\deg Q_n = n$ *and Q_n has exactly $\alpha(n,i,0)$ zeros (changes of sign) in $\Delta_{i,1}$ $(i=1, ..., m)$.*

For $i=1,2, ..., m$, we fix a monic polynomial $Q_{n,i}$, whose simple zeros coincide with the zeros of Q_n on $\Delta_{i,1}$, then

$$Q_n = \prod_{i=1}^{m} Q_{n,i}. \tag{2.1}$$

For $i=1, ..., m$, set

$$H_{n,i,0}(z) := Q_n(z) \tag{2.2}$$

$$H_{n,i,j}(z) := \int_{\Delta_{i,j}} \frac{H_{n,i,j-1}(x)}{z-x}\, d\mu_{i,j}(x), \qquad j=1, ..., b.$$

From (b) in Lemma 2.1 we know that ($k=1$, $P \equiv 1$, $g_{i,2,1} \equiv 1$)

$$H_{n,i,1}(z) = F_{n,i,1}(z). \tag{2.3}$$

LEMMA 2.3. (a) *If $b>1$, $1<j \leqslant b$, then*

$$H_{n,i,j}(z) = (-1)^{j+1} F_{n,i,j}(z) - \sum_{k=1}^{j-1} (-1)^{j+k} g_{i,k+1,j}(z)\, H_{n,i,k}(z).$$

(b) *For $i=1, ..., m$ and $j=0, 1, ..., b-1$,*

$$\int_{\Delta_{i,j+1}} H_{n,i,j}(x) \left(\sum_{k=j+1}^{b} (P_k\, g_{i,j+2,k})(x) \right) d\mu_{i,j+1}(x) = 0$$

where $P_k \in \Pi(n(i,k)-1)$ is arbitrary.

The above lemma says that $H_{n,i,j}$ has at least

$$\alpha(n,i,j) \sum_{k=j+1}^{b} n(i,k)$$

zeros (changes of sign) in $\Delta_{i,j+1}$, for $0 \leqslant j < b$ (see Section 1.1).

In the following fix a monic polynomial $w_{n,i,j}$ ($i=1, ..., m$),

$$\deg w_{n,i,j} = \sum_{k=j+1}^{b} n(i,k) := \alpha(n,i,j), \tag{2.4}$$

$j=0, 1, ..., b-1$, whose simple zeros coincide with the points where $H_{n,i,j}$ changes sign in $\Delta_{i,j+1}$ (note that $w_{n,i,0} = Q_{n,i}$).

LEMMA 2.4. *For $i=1, 2, ..., m$,*

(a) *for $0<j<b$,*

$$\int_{\Delta_{i,j}} (H_{n,i,j-1} P_j)(x) \frac{d\mu_{i,j}(x)}{w_{n,i,j}(x)} = 0$$

where P_j *is arbitrary*, $P_j \in \Pi(\alpha(n, i, j-1)-1)$.

(b) *For* $0 \leqslant j < b$, $H_{n,i,j}/w_{n,i,j}$ *does not have zeros in* $\Delta_{i,j+1}$.

(c) *For* $0 < j < b$,

$$\frac{H_{n,i,j}(z)}{w_{n,i,j}(z)} = \frac{1}{w_{n,i,j-1}(z)} \int_{\Delta_{i,j}} \frac{w^2_{n,i,j-1}(x)}{z-x} \frac{H_{n,i,j-1}(x)}{w_{n,i,j-1}(x)} \frac{d\mu_{i,j}(x)}{w_{n,i,j}(x)}.$$

The assertion (b) in Lemma 2.4 allows us to define new measures $\lambda_{i,j}$, such that polynomials $w_{n,i,j-1}$ are orthogonals with respect to $\lambda_{i,j}$. Set

$$f_{n,i,b}(x) = \left| \frac{H_{n,i,b-1}(x)}{w_{n,i,b-1}(x)} \right|$$

and

$$d\lambda_{i,b}(x) = f_{n,i,b}(x)\, d\mu_{i,b}(x)$$

and for $b > 1$, $0 < j < b$,

$$f_{n,i,j}(x) = \frac{|H_{n,i,j-1}(x)|}{|w_{n,i,j-1}(x)\, w_{n,i,j}(x)|}$$

and

$$d\lambda_{i,j}(x) = f_{n,i,j}(x)\, d\mu_{i,j}(x).$$

LEMMA 2.5. *Given* i $(i = 1, ..., m)$,

$$\int_{\Delta_{i,j}} w^2_{n,i,j-1}(x)\, d\lambda_{i,j}(x) = \inf_P \int_{\Delta_{i,j}} P^2(x)\, d\lambda_{i,j}(x)$$

for $j = 1, ..., b$, *where* $P \in \Pi^*(\deg w_{n,i,j-1})$.

We will use the above relation to obtain a problem in potential theory.

Notation. For $i = 1, ..., m$ and $j = 0, ..., 1, b-1$, denote by $\varphi_{n,i,j}$ the measure in $M_{\alpha(n,i,j)}(\Delta_{i,j+1})$ associated to $w_{n,i,j}$ (see Notation in Section 1.4). We know that $\varphi_{n,i,j} \in M(\Delta_{i,j+1})$. We must show that there exist $\varphi_{i,j} \in M(\Delta_{i,j+1})$, such that

$$\varphi_{n,i,j} \to \varphi_{i,j}, \qquad \text{as} \quad n \to \infty, \tag{2.5}$$

$i = 1, ..., m$; $j = 0, ..., b-1$ (weak convergence). Here the additional hypothesis $\mu'_{i,j} > 0$ is needed.

It is known that $M(\Delta_{i, j+1})$ is weakly compact. Therefore, each sequence $\{\varphi_{n, i, j}\}$ contains a subsequence which converges weakly to a measure $\varphi^*_{i, j}$ on $M(\Delta_{i, j+1})$; thus, in order to prove (2.5) it is sufficient to check that any family of limit measures $\varphi^*_{i, j}$ $(i=1, ..., m;\ j=0, ..., b-1)$ is the unique solution of a system of extremal equations which does not depend on the sequence involved in the selection of the limit points $\varphi^*_{i, j}$. In fact, the extremal equations (and the solution) only depends on the geometrical distribution of the segments $\Delta_{i, j}$ and certain numerical constants which describe the proportion with which interpolation is distributed along the different segments (and functions).

Following ideas of [GoRa2], the system of equations is obtained from Lemma 2.5 and

LEMMA 2.6 (see [GoRa3]). *Let K be a closed (finite) interval in* $\mathbf{R}$, $\mu \in M(K)$, $\mu' > 0$ *a.e. Let R_n be a sequence of monic polynomials with all its zeros in K*, $\deg R_n = r_n$, $r_n \to \infty$, *and h_n a sequence of functions, $h_n \neq 0$ on K, such that*

$$\frac{\log h_n}{r_n} \to h$$

uniformly in K. If

$$\int_K \frac{|R_n(s)|}{|h_n(s)|}\, d\mu(s) = \inf_{R \in \Pi^*(r_n)} \int_K \frac{|R(s)|}{|h_n(s)|}\, d\mu(s), \qquad n = 1, 2, ...,$$

then any limit point v in $M(K)$ of the sequence

$$v_n := \frac{1}{r_n} \log \frac{1}{|R_n|}$$

satisfies the extremal relation

$$\min_{x \in K} (V_v + h)(x) = \max_{\lambda \in M(K)} \min_{x \in K} (V_\lambda + h)(x).$$

Now, suppose that $\mu'_{i, j} > 0$ a.e. If $\Lambda \subset N$ and $\varphi^*_{i, j} \in M(\Delta_{i, j+1})$ are such that

$$\varphi_{n, i, j} \to \varphi^*_{i, j}, \qquad n \in \Lambda,$$

this allows us to write

$$\lim_{n \in \Lambda} |w_{n, i, j}(z)|^{1/\alpha(n, i, j)} = \exp - V_{\varphi^*_{i, j}}(z) \tag{2.6}$$

where the convergence is uniform on each compact subset of $C \backslash \Delta_{i, j}$.

Let $\varphi^*_{i,j}$ be defined as above and $V_{\varphi^*_{i,j}}$ be the associated potential, then

Lemma 2.7. (a) *For* $i=1, ..., m$,

$$\min_{x\in\Delta_{i,1}}\left(2bV_{\varphi^*_{i,0}}-(b-1)\,V_{\varphi^*_{i,1}}+b\sum_{k=1,\,k\neq i}^{m}V_{\varphi^*_{k,0}}\right)(x)$$

$$=\max_{\lambda\in M(\Delta_{i,1})}\ \min_{x\in\Delta_{i,1}}\left(2bV_\lambda-(b-1)\,V_{\varphi^*_{i,1}}+b\sum_{k=1,\,k\neq i}^{m}V_{\varphi^*_{k,0}}\right)(x)$$

where $\varphi^*_{i,1}=0$ *if* $b=1$.

(b) *If* $b>1$, *for* $i=1, ..., m$ *and* $1<j\leqslant b$,

$$\min_{x\in\Delta_{i,j}}(2(b+1-j)\ V_{\varphi^*_{i,j-1}}-(b-j)\,V_{\varphi^*_{i,j}}-(b+2-j)\,V_{\varphi^*_{i,j-2}})(x)$$

$$=\max_{\lambda\in M(\Delta_{i,j})}\ \min_{x\in\Delta_{i,j}}(2(b+1-j)\,V_\lambda-(b-j)\,V_{\varphi^*_{i,j}}-(b+2-j)\,V_{\varphi^*_{i,j-2}})(x)$$

where $\varphi^*_{i,j}=0$ *if* $j=b$.

Note that, to each segment $\Delta_{i,j}$, we associate an extremal equation and a measure $\varphi^*_{i,k}$. An important remark related to the above system is that, if we take the measures $\varphi^*_{i,k}$ as unknown, this extremal system does not depend on the measures $\mu_{i,j}$ and, using techniques from potential theory, we can prove that such a system has one and only one solution (see [GoRa1; GoRa3; and NiSo, Chap. V) if the corresponding matrix of the coefficients satisfies certain conditions.

Lemma 2.8. *The extremal problem above has one and only one solution* $\varphi_{i,j}$, $i=1, 2, ..., n$, $0\leqslant j<b$. *Moreover*, $\varphi_{n,i,j}\to\varphi_{i,j}$.

Lemma 2.9. *For* $i=1, ..., m$, *let* $\varphi_{i,j}\in M(\Delta_{i,j})$ $(j=1, ..., b)$ *be the solution of the extremal problem above and* $w_{n,i,j}$ *be defined as in* (2.4). *Then*

$$\lim |w_{n,i,j}(z)|^{1/\alpha(n,i,j)}=\exp(-V_{\varphi_{i,j}}(z)).$$

Proof of Theorem 1. Theorem 1 follows from Lemma 2.9 since

$$|Q_n(z)|^{1/n}=\prod_{i=1}^{n}|Q_{n,i}(z)|^{1/n}=\prod_{i=1}^{n}|w_{n,i,0}(z)|^{1/n}$$

$$=\exp\left\{-\sum_{i=1}^{n}\frac{\alpha(n,i,0)}{n}\,V_{\varphi_{n,i,0}}(z)\right\}$$

and

$$\frac{\alpha(n, i, 0)}{n} \to \frac{1}{m}.$$

Proof of (a) *in Theorem* 2. From (2.3) and (c) of Lemma 2.4 we have that if $b > 1$,

$$\frac{F_{n,i,1}(z)}{w_{n,i,1}(z)} = \frac{H_{n,i,1}(z)}{w_{n,i,1}(z)}$$
$$= \frac{1}{w_{n,i,0}(z)} \int_{\Delta_{i,1}} \frac{w^2_{n,i,0}(x)}{z-x} \frac{H_{n,i,0}(x)}{w_{n,i,0}(x)} \frac{d\mu_{i,1}(x)}{w_{n,i,1}(x)}.$$

We know (see (c) of Lemma 2.4) that $H_{n,i,0}(x)/w_{n,i,0}(x)$ does not change sign in $\Delta_{i,1}$; then using (b) of Lemma 2.1, we have

$$\lim \left| g_{i,1,1}(z) - \frac{P_{n,i,1}(z)}{Q_n(z)} \right|^{1/n}$$
$$= \lim \left| \frac{w_{n,i,1}(z)}{Q_n(z)\, w_{n,i,0}(z)} \int_{\Delta_{i,1}} \frac{w^2_{n,i,0}(x)\, |Q_n(x)|\, d\mu_{i,j}(x)}{|Q_{n,i}(x)\, w_{n,i,1}(x)|} \right|^{1/2}$$
$$= \exp \frac{1}{a} \left(-(b-1)\, V_{\varphi_{i,1}} + b \sum_{k=1}^{m} V_{\varphi_{k,0}} + b V_{\varphi_{i,0}} - m_{i,1} \right).$$

We will obtain an exact expression for the constant $m_{i,1}$. We observe that $w_{n,i,1}$ does not have zeros in $\Delta_{i,1}$ and $V_{\varphi_{n,i,1}} \to V_{\varphi_{i,1}}$ uniformly in $\Delta_{i,1}$. Set $s_n = \inf\{w_{n,i,1}(x) : x \in \Delta_{i,1}\}$ and $t_n = \sup\{w_{n,i,1}(x) : x \in \Delta_{i,1}\}$, then the sequence t_n/s_n is bounded.

Set

$$e_n = \{x \in \Delta_{i,1} : |(Q_n Q_{n,i})(x)| > n M_n s_n\}$$

where

$$M_n = \int_{\Delta_{i,1}} \frac{w^2_{n,i,0}(x)\, |Q_n(x)|\, d\mu_{i,1}(x)}{|Q_{n,i}(x)\, w_{n,i,1}(x)|}$$

and $E_n = \Delta_{i,1} \backslash e_n$, then

$$\mu_{i,1}(e_n)\, n M_n s_n \leqslant \int_{e_n} |(Q_n Q_{i,n})(x)|\, d\mu_{i,1}(x) \leqslant t_n M_n.$$

This says that $\mu_{i,1}(e_n) \to 0$. Taking into account that $\mu'_{i,1} > 0$ a.e., we have $|e_n| \to 0$, where $|A|$ denotes the Lebesgue measure of A.

Now we know that

$$\|Q_n Q_{n,i}\|_{\Delta_{i,1}} \leqslant \|Q_n Q_{n,i}\|_{E_n} k_n^{n+\alpha(n,i,0)}$$

where $k_n = \psi(2|\Delta_{i,1}|/|E_n| - 1) \to 1$, $\psi(x) = x + \sqrt{x^2 - 1}$, $x \geqslant 1$ (see [NiSo, Chap. V, Lemma 5.2]), so

$$\begin{aligned} M_n &\leqslant \left\| \frac{Q_n Q_{n,i}}{w_{n,i,1}} \right\|_{\Delta_{i,1}} \mu_{i,1}(\Delta_{i,1}) \leqslant \frac{1}{s_n} \|Q_n Q_{n,i}\|_{\Delta_{i,1}} \mu_{i,1}(\Delta_{i,1}) \\ &\leqslant \frac{1}{s_n} \|Q_n Q_{n,i}\|_{E_n} k_n^{n+\alpha(n,i,0)} \mu_{i,1}(\Delta_{i,1}) \leqslant n M_n k_n^{n+\alpha(n,i,0)} \mu_{i,1}(\Delta_{i,1}). \end{aligned}$$

From the relations above

$$\begin{aligned} &\lim \left| \int_{\Delta_{i,1}} \frac{w_{n,i,0}^2(x)\, |Q_n(x)|\, d\mu_{i,1}(x)}{|Q_{n,i}(x)\, w_{n,i,1}(x)|} \right|^{1/n} \\ &= \lim \left\| \frac{Q_n Q_{n,i}}{w_{n,i,1}} \right\|_{\Delta_{i,1}}^{1/n} \\ &= \exp \frac{1}{a} \left(-\min_{x \in \Delta_{i,1}} \left(-(b-1)\, V_{\varphi_{i,1}} + b \sum_{k=1}^{m} V_{\varphi_{k,0}} + b V_{\varphi_{i,0}} \right)(x) \right). \quad \blacksquare \end{aligned}$$

Proof of (b) *in Theorem* 2. Here we use that

$$\begin{aligned} &\lim \left| g_{i,1,1}(z) - \frac{P_{n,i,1}(z)}{Q_n(z)} \right|^{1/n} \\ &\qquad = \lim \left| \frac{1}{Q_n(z)\, Q_{n,i}(z)} \int_{\Delta_{i,j}} Q_{n,i}^2(x) \prod_{k=1,\, k \neq i}^{m} Q_{n,k}\, d\mu_{i,1}(x) \right|^{1/n}. \end{aligned}$$

3. Proofs of Lemmas

Proof of Lemma 2.1. The function $F_{i,1,k}$ is holomorphic in a neighborhood of infinity. Thus, for sufficiently large $R > 0$ we have (see (1.5))

$$\int_{|z|=R} F_{n,1,k}(z) = 0, \qquad j = 0, ..., n(i,k) - 1.$$

Taking into account (1.2), for $j=0, ..., n(i, k)-1$,

$$
\begin{aligned}
0 &= \int_{|z|=R} F_{n,1,k}(z)\, z^j\, dz \\
&= \int_{\Delta_{i,1}} g_{i,2,k}(x) \int_{|z|=R} \frac{Q_n(z) z^j}{z-x}\, dz\, d\mu_{i,1}(x) \\
&= 2\pi i \int_{\Delta_{i,1}} (Q_n\, g_{i,2,k})(x)\, x^j\, d\mu_{i,1}(x).
\end{aligned}
$$

From this follows (a); (b) is obtained analogously. ∎

Proof of Lemma 2.2. If $b=1$, taking into account that $g_{i,2,1}=1$, the second assertion follows immediately from (a) in Lemma 2.1. If $b>1$, we have

$$
0 = \int_{\Delta_{i,1}} Q_n(x) \sum_{k=1}^{b} (P_k\, g_{i,2,k})(x)\, d\mu_{i,1}(x), \tag{3.1}
$$

where $P_k \in \Pi(n(i, k)-1)$ is an arbitrary polynomial. Assume that Q_n has at most $\sum_{k=1}^{b} n(i, k)-1$ changes of sign on $\Delta_{i,1}$. Then we can take P_k, $k=1, ..., b$, conveniently so that

$$
\sum P_k\, g_{i,2,k}
$$

changes sign exactly at those points where Q_n does (see Section 1.1). Therefore, Q_n has at least

$$
\sum_{i=1}^{b} n(i, k)
$$

changes of sign on $\Delta_{i,1}$. Taking into account that $\deg Q_n \leqslant n$ and (1.3), we have that $\deg Q_n = n$. ∎

Proof of Lemma 2.3. (a) If $b>1$, using (2.2) and (b) of Lemma 2.1 we obtain that

$$
\begin{aligned}
H_{n,i,2}(z) &= \int_{\Delta_{i,2}} \frac{1}{z-x} \int_{\Delta_{i,1}} \frac{Q_n(s)}{x-s}\, d\mu_{i,1}(s)\, d\mu_{i,2}(x) \\
&= \int_{\Delta_{i,1}} \frac{Q_n(s)}{z-s} \int_{\Delta_{i,2}} \left(\frac{1}{z-x} - \frac{1}{s-x}\right) d\mu_{i,2}(x)\, d\mu_{i,1}(s) \\
&= g_{i,2,2}(z)\, H_{n,i,1}(z) - \int_{\Delta_{i,1}} \frac{(Q_n\, g_{i,2,2})(s)}{z-s}\, d\mu_{i,1}(s) \\
&= g_{i,2,2}(z)\, H_{n,i,1}(z) - F_{n,i,2}(z).
\end{aligned}
$$

Following the above argument, it is easy to prove that if $b>1$, $1<j\leqslant b$, then

$$H_{n,i,j}(z)=(-1)^{j+1}F_{n,i,j}(z)-\sum_{k=1}^{j-1}(-1)^{j+k}g_{i,k+1,j}(z)H_{n,i,k}(z).$$

(b) From (2.2) follows that $H_{n,i,j}$ is analytic in $C\backslash\Delta_{i,j}$ and from (2.3) and part (a) of this lemma,

$$H_{n,i,j}(z)=O(z^{-1-n(i,j)})\qquad (b>1,\ 1<j\leqslant b)$$

as $z\to\infty$.

For $j=0$, (b) is just (3.1) (see (2.3)). If $b>1$, from (a) in Lemma 2.1 and (1.2),

$$\begin{aligned}0&=\sum_{k=2}^{b}\int_{\Delta_{i,1}}(Q_n g_{i,2,k}P_k)(x)\,d\mu_{i,1}(x)\\&=\sum_{k=2}^{b}\int_{\Delta_{i,1}}(Q_nP_k)(x)\int_{\Delta_{i,2}}\frac{g_{i,3,k}(s)}{x-s}\,d\mu_{i,2}(s)\,d\mu_{i,1}(x)\\&=-\int_{\Delta_{i,2}}H_{n,i,1}(s)\sum_{k=2}^{b}(P_k g_{i,3,k})(s)\,d\mu_{1,2}(s).\end{aligned}$$

Thus (b) takes place for $j=1$.

Making in the formula above $P_2=0$ (if $b\geqslant 3$), then

$$\begin{aligned}0&=\int_{\Delta_{i,2}}H_{n,i,1}(x)\left(\sum_{k=3}^{b}P_k(x)\int_{\Delta_{i,3}}\frac{g_{i,4,k}(s)}{x-s}\,d\mu_{i,3}(s)\right)d\mu_{i,2}(x)\\&=\int_{\Delta_{i,3}}\sum_{k=3}^{b}g_{i,4,k}(s)\left(\int_{\Delta_{i,2}}\frac{H_{n,i,1}(x)}{x-s}\left(P_k(x)-P_k(s)\,d\mu_{i,2}(x)\right.\right.\\&\qquad\left.-P_k(s)\int_{\Delta_{i,2}}\frac{H_{n,i,1}(x)}{x-s}\,d\mu_{i,2}\right)d\mu_{i,3}(s)\\&=\int_{\Delta_{i,3}}H_{n,i,2}(s)\left(\sum_{k=3}^{b}g_{i,4,k}(s)P_k(s)\right)d\mu_{i,3}(s).\end{aligned}$$

For other indexes j the proof follows analogously. ∎

Proof of Lemma 2.4. Note that, if $b>1$,

$$\frac{H_{n,i,j}(z)}{w_{n,i,j}(z)},\qquad j=1,\dots,b-1,\tag{3.2}$$

is analytic on $C\backslash\Delta_{i,j}$ and the Laurent expansion of this last function at infinity starts with

$$z^{-1-\alpha(n,i,j-1)}.$$

This allows us to prove (as in Lemma 2.1) that for $j=1, ..., b-1$ (if $b>1$)

$$\int_{\Delta_{i,j}} (H_{n,i,j-1} P_j)(x) \frac{d\mu_{i,j}(x)}{w_{n,i,j}(x)} = 0$$

where P_j is arbitrary, $P_j \in \Pi(\alpha(n, i, j-1)-1)$.

We shall show that $H_{n,i,j}/w_{n,i,j}$ $(0<j<b)$ does not have zeros in $\Delta_{i,j+1}$.

From (3.2) (as in Lemma 2.1), we obtain that for $j=1, ..., b-1$,

$$\begin{aligned} \frac{H_{n,i,j}(z)}{w_{n,i,j}(z)} &= \frac{1}{w_{n,i,j-1}(z)} \int_{\Delta_{i,j}} \frac{w_{n,i,j-1}(x)}{z-x} H_{n,i,j-1}(x) \frac{d\mu_{i,j}(x)}{w_{n,i,j}(x)} \\ &= \frac{1}{w_{n,i,j-1}(z)} \int_{\Delta_{i,j}} \frac{w^2_{n,i,j-1}(x)}{z-x} \frac{H_{n,i,j-1}(x)}{w_{n,i,j-1}(x)} \frac{d\mu_{i,j}(x)}{w_{n,i,j}(x)}. \end{aligned} \tag{3.3}$$

We know that $w_{n,i,j}$ has all its zeros in $\Delta_{i,j+1}$, thus by a recursive argument, we only need to prove that $H_{n,i,1}/w_{n,i,1}$ does not change sign on $\mathbf{R}\backslash\Delta_{i,2}$. But we know that

$$\frac{H_{n,i,1}(z)}{w_{n,i,1}(z)} = \frac{1}{Q_{n,i}(z)} \int_{\Delta_{i,1}} \frac{Q^2_{n,i}(x)}{z-x} \frac{Q_n(x)}{Q_{n,i}(x)} \frac{d\mu_{i,1}(x)}{w_{n,i,1}(x)} \tag{3.4}$$

and, according to Lemma 2.2, $Q_n/Q_{n,i}$ does not have zeros in $\Delta_{i,1}$. Therefore the right-hand side of (3.4) has constant sign on $\Delta_{i,2}$, which yields our assertion.

Proof of Lemma 2.5. It is enough to rewrite the formula in (a) of Lemma 2.4.

For $b>1$, $1 \leqslant j < b$,

$$\int_{\Delta_{i,j}} w_{n,i,j}(x)\, P(x)\, d\lambda_{i,j}(x) = 0, \qquad j=1, ..., b,$$

where P is an arbitrary polynomial, $\deg P \leqslant -1 + \sum_{k=j}^{b} n(i,k)$. ∎

Proof of Lemma 2.7. We need some results.

PROPOSITION 1. *There exist an infinite set $\Lambda_1 \subset \Lambda$ and finite constants $d_{i,j}$, such that for $i=1, ..., m$, $j=1, ..., b$,*

$$\lim_{n \in \Lambda_1} \frac{1}{\alpha(n, i, j-1)} \log \left| \int_{\Delta_{i, j-1}} \frac{w_{n, i, j-2}^2(s)\, H_{n, i, j-2}(s)\, d\mu_{i, j-1}(s)}{(x-s)\, W_{n, i, j-2}(s)\, w_{n, i, j-1}(s)} \right| = d_{i, j}.$$

for $x \in \Delta_{i,j}$.

Proof. It is sufficient to prove that the sequences involved are bounded. We need the following result (see [GoRa1, *2, p. 36]): if $\sigma' > 0$ a.e. on Δ, for any sequence of monic polynomials $\{P_m\}$, with all its zeros on Δ, $\deg P_m = m$,

$$\lim_n \left(\frac{\int_\Delta |P_m|\, d\sigma}{\|P_m\|} \right)^{1/m} = 1.$$

If $j=2$ (here inf means $\inf_{x \in \Delta_{i,1}}$)

$$0 < \exp \frac{1}{a} \left\{ -\inf 2V_{\varphi_{n,i,0}}(x) + \inf \left(-\sum_{k=1, k \neq i}^{n} V_{\varphi_{n,k,0}} + V_{\varphi_{n,i,1}} \right)(x) \right\}$$

$$= \liminf_{\Lambda} \|w_{n,i,0}^2\|_{\Delta_{i,1}}^{1/n} \left(\exp \frac{1}{n} \log \inf \left| \frac{H_{n,i,0}(x)}{w_{n,i,0}(x)\, w_{n,i,1}(x)} \right| \right)$$

$$= \liminf_{\Lambda} \left(\left(\int_{\Delta_{i,j}} w_{n,i,0}^2(s)\, d\mu_{i,1}(s) \right)^{1/n} \inf \left| \frac{H_{n,i,0}(x)}{w_{n,i,0}(x)\, w_{n,i,1}(x)} \right|^{1/n} \right)$$

$$\leqslant \liminf_{\Lambda} \left| \int_{\Delta_{i,1}} \frac{w_{n,i,0}^2(s)\, H_{n,i,0}(s)\, d\mu_{i,1}(s)}{(x-s)\, w_{n,i,0}(s)\, w_{n,i,1}(s)} \right|^{1/n} \leqslant \liminf_{\Lambda} \left\| \frac{w_{n,i,0}^2 Q_n}{w_{n,i,0} w_{n,i,1}} \right\|^{1/n}$$

$$= \exp - \inf \left(\limsup_{\Lambda} \frac{1}{n} \log |w_{n,i,1}(x)| + \frac{2}{n} \log \frac{1}{|w_{n,i,0}(x)} \right|$$

$$\left. + \sum_{k=1, k \neq i}^{m} \log \frac{1}{|w_{n,i,0}(x)} \right| \Bigg)$$

$$= \exp - \frac{1}{a} \inf \left\{ -V_{\varphi_{i,1}^*}(x) + 2V_{\varphi_{i,0}^*}(x) + \sum_{k \neq i}^{m} V_{\varphi_{i,0}^*}(x) \right\} < \infty.$$

If $b > 3$ and $j = 3$, it is sufficient to note that

$$k_n \inf_{s \in \Delta_{i,2}} \frac{1}{|w_{n,i,0}(s)\, w_{n,i,2}(s)|} \inf_{t \in \Delta_{i,1}} \left| \frac{Q_n(t)}{Q_{n,i}(t)\, w_{n,i,1}(t)} \right| \|w_{n,i,1}^2\|_{\Delta_{i,2}} \|W_{n,i,0}^2\|_{\Delta_{i,1}}$$

$$\leqslant \left| \int_{\Delta_{i,2}} \frac{w_{n,i,1}^2(s)\, H_{n,i,1}(s)\, d\mu_{i,1}(s)}{(x, s)\, w_{n,i,1}(s)\, w_{n,i,2}(s)} \right|$$

$$= \left| \int_{\Delta_{i,2}} \frac{w_{n,i,1}^2(s)\, d\mu_{i,2}(s)}{(x-s)\, w_{n,i,0}(s)_{n,i,2}(s)} \int_{\Delta_{i,1}} \frac{w_{n,i,0}^2(t)\, Q_n(t)\, d\mu_{i,1}(t)}{(s-t)\, w_{n,i,0}(t)\, w_{n,i,1}(t)} \right|$$

$$\leqslant l_n \left\| \frac{w_{n,i,1}^2}{\prod_{k=0}^2 w_{n,t,k}} \right\|_{\Delta_{i,2}} \left\| \frac{Q_n w_{n,i,0}}{Q_{n,i} w_{n,i,1}} \right\|_{\Delta_{i,1}},$$

where $\lim l_n^{1/n} = \lim k_n^{1/n} = 1$.

For other indexes the proof follows analogously. ▮

Proposition 2. *Let Δ_1 be as in Proposition 1, then for $i = 1, ..., m$, $1 \leqslant j \leqslant a$, the sequence*

$$\frac{1}{\alpha(n, i, j-1)} \log f_{n,i,j}(x)$$

converges uniformly (in $\Delta_{i,j}$) to

(a) $\displaystyle \frac{b-1}{b} V_{\varphi_{i,1}^*}(x) - \sum_{k=1,\, k \neq i}^{m} V_{\varphi_{i,0}^*}(x),$ *for* $j = 1$ *if* $b > 1$.

(b) $\displaystyle \frac{b-j}{b+1-j} V_{\varphi_{i,j}^*}(x) + \frac{b+2-j}{b+1-j} V_{\varphi_{i,j-2}^*}(x) + d_{i,j}$ *if* $1 < j < b$.

(c) $2V_{\varphi_{i,b-2}^*}(x) + d_{i,b-1}$ *if* $b > 1$, $j = b$.

(d) $\displaystyle -\sum_{k=1,\, k \neq i}^{m} V_{\varphi_{i,0}^*}(x)$ *if* $b = 1 = j$.

Proof. For $j = 1$, $b > 1$,

$$\begin{aligned}
&\frac{1}{\alpha(n, i, 0)} \log f_{n,i,1}(x) \\
&\qquad = \frac{1}{\alpha(n, i, 0)} \log \left| \frac{H_{n,i,0}(x)}{w_{n,i,0}(x)\, w_{n,i,1}(x)} \right| \\
&\qquad = \frac{1}{\alpha(n, i, 0)} \log \left| \frac{Q_n(x)}{Q_{n,i}(x)\, w_{n,i,1}(x)} \right| \\
&\qquad = \frac{n}{\alpha(n, i, 0)} \left\{ \frac{\alpha(n, i, 1)}{n} V_{\varphi_{n,i,1}}(x) - \sum_{k=1,\, k \neq i}^{m} \frac{\alpha(n, k, 0)}{n} V_{\varphi_{n,k,0}}(x) \right\} \\
&\qquad\quad \to \frac{b-1}{b} V_{\varphi_{i,1}^*}(x) - \sum_{k=1,\, k \neq i}^{m} V_{\varphi_{i,0}^*}(x).
\end{aligned}$$

For $j=2,\ b\geqslant 2$,

$$\begin{aligned}
&\frac{1}{\alpha(n,i,1)}\ \log f_{n,i,2}(x)\\
&\quad=\frac{1}{\alpha(n,i,1)}\ \log\frac{|H_{n,i,1}(x)|}{|w_{n,i,1}(x)\,w_{n,i,2}(x)|}\\
&\quad=\frac{n}{\alpha(n,i,1)}\left\{\frac{\alpha(n,i,2)}{n}\ V_{\varphi_{n,i,2}}(x)+\frac{\alpha(n,i,0)}{n}\ V_{\varphi_{n,i,0}}(x)\right.\\
&\qquad\left.+\frac{1}{n}\log\left|\int_{\Delta_{i,1}}\frac{w_{n,i,0}^2(s)\,H_{n,i,0}(s)\,d\mu_{i,1}(s)}{(x-s)\,w_{n,i,0}(s)\,w_{n,i,1}(s)}\right|\right\}\\
&\qquad\to\frac{b-2}{b-1}\ V_{\varphi^*_{i,2}}(x)+\frac{b}{b-1}\ V_{\varphi^*_{i,0}}(x)+d_{i,2}.
\end{aligned}$$

For $j=3,\ b>3$,

$$\begin{aligned}
&\frac{1}{\alpha(n,i,2)}\ \log f_{n,i,3}(x)\\
&\quad=\frac{1}{\alpha(n,i,2)}\ \log\frac{|H_{n,i,2}(x)|}{|w_{n,i,2}(x)\,w_{n,i,3}(x)|}\\
&\quad=\frac{n}{\alpha(n,i,2)}\left\{\frac{\alpha(n,i,3)}{n}\ V_{\varphi_{n,i,3}}(x)+\frac{\alpha(n,i,1)}{n}\ V_{\varphi_{n,i,1}}(x)\right.\\
&\qquad\left.+\frac{1}{n}\log\left|\int_{\Delta_{i,2}}\frac{w_{n,i,1}^2(s)\,H_{n,i,1}(s)\,d\mu_{i,2}(s)}{(x-s)\,w_{n,i,1}(s)\,w_{n,i,2}(s)}\right|\right\}\\
&\qquad\to\frac{b-3}{b-2}\ V_{\varphi^*_{i,3}}(x)+\frac{b-1}{b-2}\ V_{\varphi^*_{i,1}}(x)+d_{i,3}.
\end{aligned}$$

In general, if $1<j<b$,

$$\begin{aligned}
&\frac{1}{\alpha(n,i,j-1)}\ \log f_{n,i,j}(x)\\
&\quad=\frac{1}{\alpha(n,i,j-1)}\ \log\frac{|H_{n,i,j-1}(x)|}{|w_{n,i,j-1}(x)\,w_{n,i,j}(x)|}\\
&\quad=\frac{n}{\alpha(n,i,j-1)}\left\{\frac{\alpha(n,i,j)}{n}\ V_{\varphi_{n,i,j}}(x)+\frac{\alpha(n,i,j-2)}{n}\ V_{\varphi_{n,i,j-2}}(x)\right.
\end{aligned}$$

$$+\frac{1}{n}\log\left|\int_{\Delta_{i,j-1}}\frac{w_{n,i,j-2}^2(s)\,H_{n,i,j-2}(s)\,d\mu_{i,j-1}(s)}{(x-s)\,w_{n,i,j-2}(s)\,w_{n,j-1}(s)}\right|\Bigg\}$$

$$\to\frac{b-j}{b+1-j}\,V_{\varphi^*_{i,j}}(x)+\frac{b+2-j}{b+1-j}\,V_{\varphi^*_{i,j-2}}(x)+d_{i,j-1}.$$

If $b>1$ and $j=b$,

$$\frac{1}{\alpha(n,i,b-1)}\log f_{n,i,b}(x)$$

$$=\frac{1}{\alpha(n,i,b-1)}\log\frac{|H_{n,i,b-1}(x)|}{|w_{n,i,b-1}(x)|}$$

$$=\frac{n}{\alpha(n,i,b-1)}\left\{\frac{\alpha(n,i,b-2)}{n}\,V_{\varphi_{n,i,b-2}}(x)\right.$$

$$\left.+\frac{1}{n}\log\left|\int_{\Delta_{i,b-1}}\frac{w_{n,i,b-2}^2(s)\,H_{n,i,b-2}(s)\,d\mu_{i,b-1}(s)}{(x-s)\,w_{n,i,b-2}(s)\,w_{n,i,b-1}(s)}\right|\right\}$$

$$\to 2V_{\varphi^*_{i,b-2}}(x)+d_{i,b-1}.$$

If $b=1=j$,

$$\frac{1}{\alpha(n,i,0)}\log f_{n,i,1}(x)=\frac{1}{\alpha(n,i,0)}\log\frac{|H_{n,i,0}(x)|}{|w_{n,i,0}(x)|}.$$

$$\frac{1}{\alpha(n,i,0)}\log\left|\frac{Q_n(x)}{Q_{n,i}(x)}\right|=\frac{1}{\alpha(n,i,0)}\left\{-\sum_{k=1,\,k\neq i}^{m}\frac{\alpha(n,k,0)}{n}\,V_{\varphi_{n,k,0}}(x)\right\}$$

$$\to-\sum_{k=1,\,h\neq i}^{m}V_{\varphi^*_{k,0}}(x).\quad\blacksquare$$

Now we will use Lemma 2.6: set for $i=1,\ldots,m$, $1\leqslant j\leqslant b$,

$$h_{n,i,j}=\frac{1}{f_{n,i,j}}$$

$$r_{n,i,j}=2\alpha(n,i,j-1)$$

and

$$h_{i,j}=-\tfrac{1}{2}d_{i,j},$$

where $k_{i,j}$ are defined in Proposition 1.

Taking into account Lemmas 2.5 and 2.6, we obtain that potentials $V_{\varphi^*_{i,j}}$ satisfy the extremal relations:

(I) If $j=1$ and $b>1$,

$$\min_{x\in\Delta_{i,1}}\left(V_{\varphi^*_{i,0}}-\frac{1}{2}\,\frac{b-1}{b}\,V_{\varphi^*_{i,1}}+\frac{1}{2}\sum_{k=1,\,k\neq i}^{m}V_{\varphi^*_{k,0}}-\frac{1}{2}\,k_{i,j}\right)(x)$$

$$=\max_{\lambda\in M(\Delta_{i,1})}\min_{x\in\Delta_{i,1}}\left(V_\lambda-\frac{1}{2}\,\frac{b-1}{b}\,V_{\varphi^*_{i,1}}+\frac{1}{2}\sum_{k=1,\,k\neq i}^{m}V_{\varphi^*_{k,0}}-\frac{1}{2}\,k_{i,j}\right)(x),$$

and this is equivalent to

$$\min_{x\in\Delta_{i,1}}\left(2bV_{\varphi^*_{i,0}}-(b-1)\,V_{\varphi^*_{i,1}}+b\sum_{k=1,\,k\neq i}^{m}V_{\varphi^*_{k,0}}\right)(x)$$

$$=\max_{\lambda\in M(\Delta_{i,1})}\min_{x\in\Delta_{i,1}}\left(2bV_\lambda-(b-1)\,V_{\varphi^*_{i,1}}+b\sum_{k=1,\,k\neq i}^{m}V_{\varphi^*_{k,0}}\right)(x).$$

(II) If $b>1,\ 1<j<b$,

$$\min_{x\in\Delta_{i,j}}\left(2V_{\varphi^*_{i,j-1}}-\frac{b-j}{b+1-j}\,V_{\varphi^*_{i,j}}-\frac{b+2-j}{b+1-j}\,V_{\varphi^*_{k,j-2}}\right)(x)$$

$$=\max_{\lambda\in M(\Delta_{i,j})}\min_{x\in\Delta_{i,j}}\left(2V_\lambda-\frac{b-j}{b+1-j}\,V_{\varphi^*_{i,j}}-\frac{b+2-j}{b+1-j}\,V_{\varphi^*k,\,j-2}\right)(x).$$

(III) If $b>1,\ j=b$,

$$\min_{x\in\Delta_{i,b}}(V_{\varphi^*_{i,b-1}}-V_{\varphi^*_{i,j-2}})(x)=\max_{\lambda\in M(\Delta_{i,b})}\min_{x\in\Delta_{i,b}}(V_\lambda-V_{\varphi^*_{i,j-2}})(x).$$

(IV) If $b=1=j$,

$$\min_{x\in\Delta_{i,1}}\left(2V_{\varphi^*_{i,0}}+\sum_{k=1\ k\neq i}V_{\varphi^*_{k,0}}\right)(x)$$

$$=\max_{\lambda\in M(\Delta_{i,1})}\min_{x\in\Delta_{i,1}}\left(2V_\lambda+\sum_{k=1,\,k\neq i}^{m}V_{\varphi^*_{k,0}}\right)(x).\quad\blacksquare$$

Proof of Lemma 2.8. Let us give a convenient enumeration for the measures $\varphi^*_{i,j}$ and intervals $\Delta_{i,j}$ (for convenience we introduce the extra factor $b/(b+1-j)$ in (b) of Lemma 2.7 for $1<j\leqslant b$). We enumerate the equations of Lemma 2.7 in the following way: first, the m equations in (a); second, the m equations in (b) with $j=2$; then the m equations in (b) with $j=3$, and so on.

If p is an integer then $p'=[p/m]$, where $[x]$ denotes the greatest integer q such that $q\leqslant x$.

Now for $i=1, ..., a$, set

$$\psi_i=\varphi^*_{i-i'm,\, i'} \qquad \text{and} \qquad \Delta_i=\Delta_{i-i'm,\, i'+1}; \tag{3.5}$$

so we have a measure and a intervals.

Take, for $i, j=1, ..., a$ $(j\geqslant i)$,

$$c_{i,j}:=\left\{\begin{array}{ll} 2\dfrac{(b-i')^2}{b} & \text{if} \quad j=i \\ b & \text{if} \quad 1\leqslant i, j\leqslant 1 \text{ and } j\neq i \\ -\dfrac{(b-i')(b-i'-1)}{b} & \text{if} \quad j=m+i \\ 0 & \text{otherwise} \end{array}\right\}$$

and $c_{i,j}=c_{j,i}$ for $i>j$.

Now the system in Lemma 2.7 can be written as

$$\min_{x\in\Delta_i}\sum_{j=1}^{a} c_{i,j}V_{\psi_j(x)}=\max_{\lambda\in M(\Delta_i)}\min_{x\in\Delta_i}\left(\sum_{j=i,\, j\neq i}^{a} c_{i,j}V_{\psi_j}(x)+c_{i,i}V_\lambda(x)\right)$$

for $i=1, ..., a$.

To prove that this last system has one and only one solution $\{\psi_i : i=1, ..., a\}$, it is sufficient to show that the matrix $C=(c_{i,j})$ satisfies the following conditions (see [NiSo, Chap. V]):

(I) C is symmetric;

(II) if $c_{i,j}<0$, then $\Delta_i\cap\Delta_j=\varnothing$; (3.6);

(III) C is positively defined.

It is easy to see that C is symmetric. If $c_{i,j}<0$, with $i>j$, then $j=i-m$. Taking into account that $(i-m)'=i'-1$ and

$$j-j'm=i-m-(i-m)'m=i-m-(i'-1)m=i-i'm,$$

we have

$$\Delta_i=\psi_{i-i'm,\, i'+1} \qquad \text{and} \qquad \Delta_j=\psi_{i-i'm,\, i'},$$

then (see (1.1))

$$\Delta_i\cap\Delta_j=\varnothing.$$

This proves (II) (C is a symmetric matrix).

Set

$$A=\{(i,j) : i\leqslant j,\ c_{i,j}<0\}.$$

Suppose that $x_1, ..., x_n$ are arbitrary real numbers, then

$$\begin{aligned}
&\sum_{i, j=1}^{a} c_{i, j} x_i x_j \\
&\quad = 2 \sum_{(i, j) \in A} c_{i, j} x_i x_j + \sum_{i=1}^{a} c_{i, i} x_i^2 + 2 \sum_{i, j=1,\ i<j}^{m} c_{i,} \ x_i x_j \\
&\quad = \sum_{j=1}^{b-1} \sum_{i=1}^{m} c_{(j-1)m+i,\ jm+i} x_{(j-1)m+i} x_{jm+i} \\
&\qquad + \frac{2}{b} \sum_{i=1}^{a} (b-i')^2 x_i^2 + b \left(\sum_{i=1}^{m} x_i \right)^2 - b \sum_{i=1}^{m} x_i^2 \\
&\quad = \sum_{j=1}^{b=1} \sum_{i=1}^{m} \frac{(b+1-j)(b-j)}{b} (x_{(j-1)\, m+i} - x_{jm+1})^2 \\
&\qquad - \frac{2}{b} \sum_{i=1}^{a} (b-i')^2 x_i^2 + \sum_{i=1}^{m} \frac{b(b+1)}{b} x_i^2 \\
&\qquad + \frac{2}{b} \sum_{i=1}^{a} (b-i')^2 x_i^2 + b \left(\sum_{i=1}^{m} x_i \right)^2 - b \sum_{i=1}^{m} x_i^2 \\
&\quad = \sum_{j=1}^{b-1} \sum_{i=1}^{m} \frac{(b+1-j)(b-j)}{b} (x_{(j-1)m+i} - x_{jm+i})^2 \\
&\qquad + b \left(\sum_{i=1}^{m} x_i \right)^2 + \sum_{i=1}^{m} x_i^2 \geqslant 0. \quad \blacksquare
\end{aligned}$$

Proof of Lemma 2.9. Fix i and j; we know that $\varphi_{n, i, j} \to \varphi_{i, j} \in M(\Delta_{i, j+1})$. Taking into account that

$$\deg \omega_{n, i, j} = \alpha(n, i, j)$$

and

$$|w_{n, i, j}(z)|^{1/(\alpha(n, i, j))} = \exp - \frac{1}{\alpha(n, i, j)} \log \frac{1}{w_{n, i, j}(z)} = \exp - V_{\varphi_{n, i, j}}(z),$$

we have the assertion. $\blacksquare$

4. A Special Case

Now we discuss Theorem 3. The additional conditions on $g_{1, 2, 2}$ mean that there exist real numbers α, β such that $g_{1, 2, 2} - \alpha z - \beta$ does not vanish in $C \backslash \Delta_{1,2}$. Then its reciprocal is a Markov function with measure supported in $\Delta_{1,2}$ (see the Appendix in [KrNu]). Indeed, there exists a signed measure μ^* such that

$$\frac{1}{g_{1,2,2}(z)-\alpha z-\beta}=\int_{\Delta_{1,2}} \frac{d\mu^*(s)}{z-s}, \qquad z \in C \backslash \Delta_{1,2}.$$

Hence

$$\begin{aligned}
&\int_{\Delta_{1,2}} P(s)((\alpha s+\beta) F_{n,1,1}(s)-F_{n,1,2}(s))\, d\mu^*(s) \\
&\quad=\int_{\Delta_{1,1}} (PQ_n)(x)(\alpha x+\beta-g_{1,2,2}(x)) \int_{\Delta_{1,2}} \frac{d\mu^*(s)}{x-s}\, d\mu_{1,1}(x) \\
&\quad=\int_{\Delta_{1,1}} (PQ_n)(x)\, d\mu_{1,1}(x)=0,
\end{aligned}$$

where $P \in \Pi(n(1,1)-1)$ is an arbitrary polynomial. Consequently,

$$(\alpha s+\beta) F_{n,1,1}(s)-F_{n,1,2}(s)$$

has at least $n(1,1)$ zeros in $\Delta_{1,2}$. Let us prove that there exist no other zeros. If for some polynomial R, $\deg R \geqslant n(1,1)+1$, with zeros in $\Delta_{1,2}$,

$$\frac{(\alpha z+\beta) F_{n,1,1}(z)-F_{n,1,2}(z)}{R(z)}$$

is holomorphic in $C \backslash \Delta_{1,1}$ then

$$\int_{\Delta_{1,1}} Q_n(x)\, P(x)(\alpha x+\beta-g_{1,2,2}(x)) \frac{d\mu_1(x)}{R(x)}=0$$

for $P \in \Pi(n(1,1)+n(1,2)-1)$ arbitrary. This says that Q_n has at least $n(1,1)+n(1,2)+1$ zeros in $\Delta_{1,1}$ and this is not possible (see Lemma 2.2). We have obtained the following

Proposition 3. *There exists a polynomial $w_{n,2}^* \in \Pi^*(n(1,1))$ such that all zeros of $w_{n,2}$ lie in $[c,d]$ and*

$$\begin{aligned}
&\int_{\Delta_{1,2}} |w_{n,2}^*(s)((\alpha s+\beta F_{n,1,1}(s)\, F_{n,1,2}(s)|\, d\mu^*(s) \\
&\quad=\min_{q \in \Pi^*(n(1,1))} \int_{\Delta_{2,1}} |q^2(s)| \frac{|(\alpha s+\beta) F_{n,1,1}(s)-F_{n,1,2}(s)|\, d\mu^*(s)}{|w_{n,2}^*(s)|}
\end{aligned}$$

Using this extremal relation, reasoning as in the Proof of Theorems 1 and 2, we can obtain the result announced in Theorem 3. Taking $g_{2,2,2}$ in place of $g_{1,2,2}$, we obtain an analogous result.

REFERENCES

[A] A. ANGELESCO, Sur l'approximation simultane de plusieurs intégrales defines, *C.R. Acad. Sci. Paris* **167** (1918), 629–631.

[ApSt] A. I. APTEKAREV AND H. STAHL, Asymptotics of Hermite–Pade polynomials, *in* "Progress in Approximation Theory" (A. A. Gonchar and E. B. Saff, Eds.), pp. 127–167, Springer-Verlag, New York/Berlin, 1992.

[BuLo1] J. BUSTAMANTE AND G. LOPEZ LAGOMASINO, Hermite–Pade approximation to a Nikishin type system of analytic functions, *Mat. Sb.* **183**, No. 11 (1992), 117–138.

[BuLo2] J. BUSTAMENTE AND G. LOPEZ LAGOMASINO, Hermite–Pade approximation to a Nikishin type system of meromorphic functions, *Approx. Theory* **81**, No. 1 (1995), 1–12.

[DriSt] K. DRIVER AND H. STAHL, Normality in Nikishin systems, *Indag. Math. N. S.* **5** (2) (1994), 161–187.

[GoRa1] A. A. GONCHAR AND E. A. RAKHMANOV, On the convergence of simultaneous Pade approximants for a system of functions of Markov type, *in* "Proc. Steklov Inst. Math." **3** (1983), 31–50,

[GoRa2] A. A. GONCHAR AND E. A. RAKHMANOV, Equilibrium distribution and the rate of rational approximation of analytic functions, *Math. USSR Sb.* **62** (1989), 305–348.

[GoRa3] A. A. GONCHAR AND E. A. RAKHMANOV, On an equilibrium problem for vector potentials, *Uspekhi Mat. Nauk* **40**, No. 4 (244), (1985), 155–156.

[KrNu] M. G. KREIN AND A. A. NUDELMAN, "The Markov Moment Problem and Extremal Problems," Amer. Math. Soc., Providence, RI, 1977.

[Lan] N. S. LANDKOV, "Foundations of Modern Potential Theory," Springer-Verlag, New York/Berlin, 1972.

[Ni] E. M. NIKISHIN, On simultaneous Pade approximants, *Math. USSR Sb.* **41** (1982), 402–425.

[NiSo] E. M. NIKISHIN AND V. N. SOROKIN, "Rational Approximation and Orthogonality," Trans. of Math. Monographs, Vol. 1992, American Math. Society, Providence, RI, 1991.

[Pi] L. R. PINEIRO, On simultaneous approximants for some set of Markov functions, *Moscow Univ. Mat. Bull.* **2** (1987), 67–70.